Total Curvature and Packing of Knots

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Abstract

It is well known that some complicated knots can have small total curvature: in particular, there are families of knots whose crossing numbers grow arbitrarily large, while the total curvatures remain uniformly bounded. The most familiar examples involve braids that look long and thin. In this paper, we show that the heuristic description "long and thin" is indeed the only way to have bounded total curvature: If K is a smooth knot in $\mathbb{R}^3$, R the cross-section radius of a uniform tube neighborhood K, L the arclength of K, and κ the total curvature of K, then

$$\text{crossing number of } K \leq \frac{L}{R}\,\kappa\,.$$

There exist families of knots in which the crossing numbers grow as fast as the (4/3) power of $\frac{L}{R}$. Our theorem says that such families have unbounded total curvature: If the total curvature is bounded, then the growth rate can only be linear.

On the way to this theorem, we establish fundamental lemmas about packing curves in limited volume: If a long smooth curve K with arclength L is contained in a solid ball of radius ρ, then the total curvature of K is at least proportional to L/ρ. If a long smooth curve K is contained in a spherical shell of radii $a < b$, then the total curvature of K is at least proportional to $1/\sqrt{a}$.

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1 Introduction

The total curvature of smooth closed curve in $\mathbb{R}^3$ must be at least 2π; this is the classical theorem of Fenchel [5, 11]. If the curve actually is a nontrivial knot, then the Fary-Milnor theorem [6, 11] says the total curvature must be $> 4\pi$. Are there properties of the knot that could guarantee larger total curvature? Successive composition [7] or or other kinds of satellite constructions ([12] together with [11]) work. On the other hand, topological complexity in the form of high crossing-number is not enough: it is well known since [10] that one can construct knots with arbitrarily large minimum crossing-number represented by curves with uniformly bounded total curvature. Here is one way to build them.

Example 1 (Knots with bounded total curvature). *Fix any odd integer n. Construct a smooth knot K_n, with minimum crossing number n, as the union of four arcs H, A, C_1, C_2, where the total curvatures are $\kappa(H) \to 0$ as $n \to \infty$, $\kappa(A) = 0$, and $\kappa(C_1) \approx \kappa(C_2) \approx 2\pi$. Let H be the circular helix in $\mathbb{R}^3$ parametrized as $[\cos(t), \sin(t), n^2 t]$, $t = 0 \dots n\pi$.* The height coordinate $n^2 t$ makes $\kappa(H)$ behave like $1/n$ for large n. Using any exponent larger than 1, i.e. $n^{1+\epsilon} t$, still makes $\kappa(H) \to 0$. *Let A be the central axis of the cylinder on which H runs. Let C_1 and C_2 be curves that smoothly connect the top of H to the bottom of A and vice-versa.* For large n, the tangent vectors at the beginning and end of H are nearly vertical. The arcs C_1 and C_2 can be chosen to be almost planar-convex curves, with total curvatures $\kappa(C_i) \approx 2\pi$. Similarly, *for any (p, q), torus knots or links of type (p meridians, q longitudes), can have total curvature close to $2\pi q$ if they are drawn on a standard torus that is long and thin enough.*

In this paper, we show that examples of the preceeding kind are, in a sense, the only kind possible. In order to have total curvature uniformly bounded, the knots must be "long and thin"; if we imagine the knots being made of actual "rope", then the ratio of length to rope-thickness is large.

Definition 1. *Suppose K is a smooth knot in $\mathbb{R}^3$. For $r > 0$, consider the disks of radius r normal to K, centered at points of K. For r sufficiently small, these disks are pairwise disjoint and combine to form a tubular neighborhood of K. Let $R(K)$, the* thickness radius *of K, denote the supremum of such "good" radii. The* ropelength *of K, denoted $E_L(K)$, is the ratio*

$$E_L(K) = \frac{\text{total arclength of } K}{R(K)} .$$

Definition 2. *Let K be a smooth knot. From almost every direction, if we project K into a plane, the projection is regular, in particular there are only finitely many crossings. We can average this crossing-number over all directions of projection (i.e. over the almost-all set of directions that give regular projections). This* average crossing number *is denoted* $\mathrm{acn(K)}$*.* Certainly, the minimum crossing-number of the knot-type, $\mathrm{cr[K]}$, satisfies $\mathrm{cr[K]} \leq \mathrm{acn(K)}$.

Our main result is the following:

Theorem 1. *If K is a smooth knot in $\mathbb{R}^3$, then (up to a multiplicative constant independent of K)*

$$\mathrm{acn(K)} \leq \mathrm{E_L(K)}\ \kappa(\mathrm{K})\ .$$

In particular, if we are given some family of knots in which total curvature is uniformly bounded, while crossing number is growing, then the ropelength must be growing at least as fast as the crossing numbers. Alternatively, if the crossing numbers are growing faster than ropelength, then the total curvatures must be growing fast enough to make up the difference. We showed in [2, 3] that $\mathrm{acn(K)} \leq \mathrm{E_L(K)}^{4/3}$, and there are examples [1, 4] where the 4/3 power is achieved. In the particular examples of [1, 4], the knots and links have evident growing total curvature; our theorem says that some unbounded amount of total curvature must occur in any situation of more-than-linear growth of crossings with ropelength.

If we model a knot made of actual "rope" as a smooth curve with a uniform tube neighborhood, then the thickness (radius) r of that rope is $\leq R(K)$, so $E_L(K) \leq \frac{L}{r}$. Thus the theorem also holds with $\frac{L}{r}$ in place of E_L.

2 Lemmas on total curvature

The three lemmas in this section establish fundamental properties of smooth space-curves, relating total curvature to packing, to oscillation relative to a given point, and to the "illumination" of a given point. We deal in this section with smooth space-curves, not assuming the curves are simple or closed; and we make no use of thickness.

To keep the arguments as simple as possible, we assume throughout the paper that "smooth" means smooth of class C^2. For a smooth curve A, we denote the total curvature of A by $\kappa(A)$.

It is intuitively clear that if a long rope is packed in a small box, then the rope must curve a lot. This fundamental lemma is an important ingredient in our analysis of the interplay between ropelength, crossing number, and total curvature.

Lemma 1.1 (Packing and curvature). *Suppose A is a smooth connected curve of length L, contained in a round 3-ball of radius ρ , where $L \geq 3\rho$. Then $\kappa(A)$ is approximately proportional to at least L/ρ. More precisely,*

$$L \geq 3\rho \implies \kappa(A) \geq 1/3 \; , \tag{1}$$

and, in general,

$$\kappa(A) \geq \frac{1}{3}\, n \;, \text{ where } n \text{ is the greatest integer } n = \left[\frac{1}{3}\frac{L}{\rho}\right] \; .$$

Saying this without "n", we have

$$\frac{L}{\rho} \leq 9\,\kappa(A) + 3 \; .$$

Proof. First consider the case of a smooth connected arc A, of length $L \geq 3$ contained in a ball B of radius $\rho = 1$. We want to show the total curvature $\kappa(A) \geq \frac{1}{3}$.

Let $t \to x(t)$ be unit-speed parametrization of A, where $x(0)$ is one endpoint of A and $x(L)$ the other. The fundamental theorem of calculus, applied first to $x(L)$ and then again to $x'(s)$, says

$$x(L) = x(0) + \int_{s=0}^{L} x'(s)\; ds \;=\; x(0) + Lx'(0) + \int_{s=0}^{L}\int_{u=0}^{s} x''(u)\; du\; ds \; . \tag{2}$$

The point $x(0)+Lx'(0)$ lies well outside the ball B, and the vector $\int_{s=0}^{L}\int_{u=0}^{s} x''(u)\; du\; ds$ has to be long enough to get back in. Specifically, we must have

$$L - 2 \leq \left|\int_{s=0}^{L}\int_{u=0}^{s} x''(u)\; du\; ds\right| \leq \int_{s=0}^{L}\int_{u=0}^{s} |x''(u)|\; du\; ds \tag{3}$$

$$= \int_{s=0}^{L} \kappa(A_{[0,s]})\; ds \;<\; \int_{s=0}^{L} \kappa(A) = L \cdot \kappa(A) \; . \tag{4}$$

Thus

$$1 - \frac{2}{L} < \kappa(A) \ .$$

Since $L \geq 3$, we have $\kappa(A) > 1 - \frac{2}{3}$.

Next consider the case of a connected arc A of length $L \geq 3\rho$ contained in a ball of radius ρ. We want again to prove that $\kappa(A) \geq \frac{1}{3}$. Rescale the problem by multiplying all coordinates by $1/\rho$. By the first case, the resulting arc has total curvature $\geq 1/3$. But rescaling does not change total curvature.

Finally, consider the general case. Partition the arc A into n arcs, each having length $\geq 3\rho$, with perhaps one smaller arc left over, and apply the previous case to each of the n arcs. □

Another way that a long curve is forced to have a lot of total curvature is if its distance from some given point oscillates a great deal. This is captured in the next lemma.

Lemma 1.2 (Oscillation and curvature). *Let S_a, S_b be concentric spheres with radii $a < b$. Let A be a smooth curve contained in the spherical shell bounded by S_a and S_b. Suppose the two endpoints of A lie on S_a and at least one point of A lies on S_b. Then the total curvature is at least on the order of $1/\sqrt{a}$. More precisely,*

$$\kappa(A) \geq \pi - 2 \arcsin(a/b) \ . \tag{5}$$

If $b \geq a + 1$, then

$$\kappa(A) > \frac{2\sqrt{2}}{\sqrt{a+1}} \ . \tag{6}$$

If $a \geq 2$, the bound (6), along with simplifying the coefficient, gives

$$\kappa(A) > \frac{2}{\sqrt{a}} \ . \tag{7}$$

Proof. We shall reduce the proof of (5) to a standard trig exercise, and then calculate (6).

Let p, q be the endpoints of A, z a point of $A \cap S_b$, $\bar{pz}$ and $\bar{zq}$ the line segments from p to z and from z to q, and let P be the two-edge polygon $\bar{pz} \cup \bar{zq}$. Since the polygon P is an inscribed polygon of A, we know from [11] that $\kappa(A) \geq \kappa(P)$, so it suffices to establish the desired lower bound for $\kappa(P)$. If either of the edges of P is not tangent to the sphere S_a, we can pivot the edge at point z to move the edge to tangency in a way that opens the angle $\widehat{pzq}$, so reducing $\kappa(P)$. Thus, it suffices to prove the lower bound for tangent two-edge polygons. The three points p, z, q determine a plane; we wish that plane also would include the center of the spheres. If not, then (keeping the two edges tangent to S_a, and allowing the points of tangency and the angle $\widehat{pzq}$ to change), rotate the plane of p, z, q (with axis of rotation the line through z parallel to $\bar{pq}$) until it does contain the center of the spheres. This deformation also would increase the angle $\widehat{pzq}$, and so decrease the total curvature of P. Thus, we are reduced to the situation where p, z, q and the center of the spheres are coplanar, and P consists of tangent lines to S_a drawn symmetrically from point z on S_b. We then have a right-triangle with

$$\sin(\frac{\widehat{pzq}}{2}) = \frac{a}{b} \ ,$$

which gives (5).

To derive (6), first note that $\pi - 2\text{arcsin}\ (a/b)$ increases as $(b-a)$ gets larger. So if we show $\pi - 2\text{arcsin}\ (a/b) \geq \frac{2\sqrt{2}}{\sqrt{a+1}}$ for $b = a+1$, then we will have that inequality for all $b \geq a+1$.

Rewrite (5) in terms of a and $b = a+1$,

$$\kappa(K) \geq \pi - 2\arcsin\left(1 - \frac{1}{a+1}\right) \ . \tag{8}$$

Now let $t = \sqrt{\frac{1}{a+1}}$ and check (analytically or graphically) that

$$\pi - 2\arcsin(1-t^2) > (2\sqrt{2})\ t \ . \tag{9}$$

$\square$

In the next lemma, we call the curve Y instead of A, to help clarify how the lemmas will be used. We prove Lemma 1.3 by applying Lemmas 1.1 and 1.2 to subarcs A of Y.

Suppose Y is a smooth curve in $\mathbb{R}^3$, and x_0 is a point some finite distance from Y. The integral

$$\int_{y\in Y} \frac{1}{|y-x_0|^2} \tag{10}$$

can be thought of as measuring the "illumination" of x_0 by Y.

Lemma 1.3 (Illumination and curvature). *Suppose Y is a smooth curve in $\mathbb{R}^3$, and x_0 is a point such that $\forall y \in Y$, $|y - x_0| \geq 2$. Then the illumination of x_0 by Y is bounded by the total curvature of Y. More precisely,*

$$\int_{y \in Y} \frac{1}{|y - x_0|^2} \leq c + 66\, \kappa(Y) \;, \tag{11}$$

where c is a universal constant independent of Y.

Lemma 1.3 is perhaps the most intricate part of the paper. Before proving it, we present four special cases. The general argument does not reduce to these special cases – rather we include them to give an intuitive sense of why the proposition might be true (the first three), and some of issues one needs to confront in building a proof (the fourth).

2.1 Special cases for Lemma 1.3

2.1.1

Suppose Y is a straight line, starting at a point 2 units from x_0 and aiming radially away from x_0. Then the line integral is just $\int_2^\infty 1/s^2\, ds = 1/2$. □

2.1.2

Suppose Y is a straight line, infinite in both directions, and tangent to the sphere of radius 2 centered at x_0. Then

$$\int_{y \in Y} \frac{1}{|y - x_0|^2} = \int_{-\infty}^{\infty} \frac{1}{4 + s^2}\, ds \quad = \quad \frac{\pi}{2} \;.$$

□

2.1.3

Suppose Y is a polygonal path (or closed curve) consisting of e edges (of possibly varying lengths), such that each pair of consecutive edges meets at a right angle. Then, by the second special case, each edge of Y contributes $< \pi/2$ to the illumination integral, so

$$\begin{aligned} \int_{y \in Y} \frac{1}{|y - x_0|^2} < e\, \frac{\pi}{2} &= \kappa(Y) \quad \text{if the polygon has endpoints} \\ &= \kappa(Y) + \pi/2 \quad \text{if the polygon is closed.} \end{aligned}$$

□

2.1.4

Suppose Y is a smooth curve, starting at a point y_0 with $|y_0 - x_0| = 2$, with the property that the distance function $|y - x_0|$ is monotone increasing on Y.

For $n = 2, 3, \ldots$, let $B[n]$ denote the round ball of radius n centered at x_0, and let $S[n, n+1]$ denote the spherical shell with radii n and $n+1$. By our assumption of monotonicity, each intersection $Y \cap S[n, n+2]$ is a connected arc, which we denote Y_n. Let $\ell[n, n+1]$ denote the arclength of Y_n. Then

$$\int_{y \in Y} \frac{1}{|y - x_0|^2} = \sum_{n=2}^{\infty} \int_{y \in Y_n} \frac{1}{|y - x_0|^2} \leq \sum_{n=2}^{\infty} \frac{\ell(Y_n)}{n^2} \ .$$

We would like to bound all the numbers $\ell(Y_n)$, in terms of total curvature of Y, somehow using Lemma 1.1. That lemma gives upper bounds for the lengths $\ell(Y \cap B[n])$ in terms of total curvature. But we cannot go directly from that to a simple bound on the lengths in the individual shells: For given curve Y, it is possible to satisfy Lemma 1.1 by having the lengths contained in inner shells relatively small, and then a lot of Y is packed into some outer shell. We get around this problem by bounding (not the illumination integral from Y itself, but rather) the illumination integral of a hypothetical curve Y^* that is packed around x_0 in such a way as to make the illumination integral as large as possible subject to the constraints imposed by Lemma 1.1. (In this intuitive discussion of the special case, we will continue with the image of a "hypothetical curve". In the actual proof of Lemma 1.3, we will be more rigorous.)

For brevity, let κ denote $\kappa(Y)$. By our assumptions on the shape of Y relative to x_0, we start with $Y_2 = Y \cap B[3]$. By Lemma 1.1,

$$\ell(Y \cap B[3]) \leq 3(9\,\kappa + 3) \ .$$

Similarly,

$$\ell(Y \cap B[4]) \leq 4(9\,\kappa + 3) \ ,$$
$$\ell(Y \cap B[5]) \leq 5(9\,\kappa + 3) \ , \text{etc.}$$

If the curve Y does not actually achieve these bounds, then add extra length (the "hypothetical" curve) in each of the shells as needed to actually reach these bounds. Since we are adding length to the Y that already exists, the illumination integral can only increase. Thus the illumination of Y^* is an upper bound for the illumination of Y.

We have

$$\ell(Y^* \cap B[3]) = 3(9\,\kappa + 3)\ ,$$
$$\ell(Y^* \cap B[4]) = 4(9\,\kappa + 3)\ ,$$
$$\ell(Y^* \cap B[5]) = 5(9\,\kappa + 3)\ , \text{etc.}$$

Thus

$$\ell(Y^* \cap S[2,3]) = 3(9\,\kappa + 3)\ ,$$
$$\ell(Y^* \cap S[3,4]) = 4(9\,\kappa + 3)\ -\ 3(9\,\kappa + 3)\ =\ (9\,\kappa + 3)\ ,$$
$$\ell(Y^* \cap S[4,5]) = 5(9\,\kappa + 3)\ -\ 4(9\,\kappa + 3)\ =\ (9\,\kappa + 3)\ , \text{etc.}$$

And so,

$$\int_{y \in Y} \frac{1}{|y - x_0|^2} \leq \int_{y \in Y^*} \frac{1}{|y - x_0|^2} < \frac{3(9\,\kappa + 3)}{2^2}\ +\ \sum_{n=3}^{\infty} \frac{(9\,\kappa + 3)}{n^2} < 13\kappa + 5\ .$$

□

Proof. (of Lemma 1.3)

□

3 Proof of Theorem 1

As a preliminary step, rescale the knot so the thickness radius $R(K) = 1$. This has no effect on the total curvature or on the average crossing number, and simplifies the ratio $E_L(K)$ to just the length, L. We want to show

$$\mathrm{acn(K)} < \mathrm{c} \cdot \mathrm{L} \cdot \kappa(\mathrm{K})\ , \tag{12}$$

where c is some coefficient that works for all knots.

The average crossing number of a knot can be expressed as an integral over the knot [8], similar to Gauss's double integral formula for the linking number of two loops. Specifically,

$$\mathrm{acn(K)} = \frac{1}{4\pi} \int_{\mathrm{x} \in \mathrm{K}} \int_{\mathrm{y} \in \mathrm{K}} \frac{|< \mathrm{T_x}\,, \mathrm{T_y}\,, \mathrm{x} - \mathrm{y} >|}{|\mathrm{x} - \mathrm{y}|^3}\ , \tag{13}$$

where T_x, T_y are the unit tangents at x, y and $< u, v, w >$ is the triple scalar product $(u \times v) \cdot w$ of the three vectors u, v, w.

Omit the ubiquitous $\frac{1}{4\pi}$ (the first contribution to "c") and write the integral as a sum of two terms:

$$\mathrm{Near}(K) = \int_{x\in K}\int_{\mathrm{arc(x,y)}\leq\pi} \frac{|<T_x\,,T_y\,,x-y>|}{|x-y|^3} \quad ,$$

and

$$\mathrm{Far}(K) = \int_{x\in K}\int_{\mathrm{arc(x,y)}\geq\pi} \frac{|<T_x\,,T_y\,,x-y>|}{|x-y|^3} \quad .$$

We shall analyze these contributions separately, and find bounds of the form

$$\mathrm{Near}(K) \leq c_1\, L \tag{14}$$

$$\mathrm{Far}(K) \leq c_2\, L\, \kappa(K)\ . \tag{15}$$

In the first case, we obtain a uniform constant bound (independent of curve K and choice of point x) for the inner integral, so

$$\mathrm{Near}(K) \leq (L)(\text{that uniform bound})\ .$$

In the second case, we obtain a bound for the inner integral of the form

$$(\text{some uniform constant})(\text{total curvature of } K)\ ,$$

so

$$\mathrm{Far}(K) \leq (L)(\text{that uniform constant})\cdot\kappa(K)\ .$$

3.1 Bounding Near(K)

Proof. We shall show that the inner integral is uniformly bounded, independent of K.

For any smooth curve with thickness radius R, it is shown in [9] that the curvature at each point is at most $1/R$. So in the present situation, we know that the curvature of K is everywhere ≤ 1.

Let $\theta \to x(\theta)$ be a unit speed parametrization of K. So $x'(\theta) = T_x$ and $|x''(\theta)| \leq 1$. We are studying points y for which $\mathrm{arc}(x,y) \leq \pi$, so we can take for the parameter set the interval $[0,\pi]$, with our starting point $x = x(0)$ and $y = y(\theta)$ for some $\theta \in [0,\pi]$. Using the same parameter set, let $\theta \to p(\theta)$ be an arclength preserving parametrization of the unit semi-circle. Since the curvature of K is everywhere bounded by the curvature of the unit circle, Schur's theorem [5] tells us that for each θ,

$$|x(\theta) - x(0)| \geq |p(\theta) - p(0)| \;, \tag{16}$$

That is,

$$|y - x| \geq \sqrt{2 - 2\cos\theta} \;. \tag{17}$$

Thus

$$\frac{|< T_x \,, T_y \,, x - y >|}{|x-y|^3} \leq \frac{|< T_x \,, T_y \,, \frac{x-y}{|x-y|} >|}{2 - 2\cos\theta} \tag{18}$$

$$= \frac{|< T_x \,, T_y \,, \frac{x-y}{\theta} >|}{2 - 2\cos\theta} \;\; \frac{\theta}{|x-y|} \tag{19}$$

Using Schur's theorem again, we have $|x - y| \geq |p(\theta) - p(0)| = \sqrt{2 - 2\cos\theta}$. The function $\frac{\theta}{\sqrt{2-2\cos\theta}}$ is increasing on $[0, \pi]$, with maximum value $\pi/2$. So

$$\frac{|< T_x \,, T_y \,, x - y >|}{|x-y|^3} \leq \frac{\pi}{2} \; \frac{|< T_x \,, T_y \,, \frac{x-y}{\theta} >|}{2 - 2\cos\theta} \;. \tag{20}$$

The vectors T_y and $\frac{x-y}{\theta}$ are each first-order (in terms of θ) close to T_x. Specifically, we have for T_y,

$$T_y = T_x + \int_{t=0}^{\theta} x''(t) \;. \tag{21}$$

Since $|x''| \leq 1$, this says we can write T_y as $T_x + V$, where $|V| \leq \theta$. Meanwhile, (2) says we can write $\frac{|x-y|}{\theta}$ as $T_x + W$, where $|W| \leq \frac{1}{2}\theta$. Thus

$$T_x \times T_y = T_x \times V \;,$$

a vector perpendicular to T_x with length $\leq \theta$. When we take the dot product of this vector with $T_x + W$, we just get the dot product with W, so a number whose size is at most $\frac{1}{2}\theta^2$.

We now have

$$\frac{|< T_x \,, T_y \,, x - y >|}{|x-y|^3} \leq \frac{\pi}{4} \;\; \frac{\theta^2}{2 - 2\cos\theta} \;\; \leq \frac{\pi}{4} \left(\frac{\pi}{2}\right)^2 \;, \tag{22}$$

so the inner integral is bounded by $(2\pi) \cdot (\frac{\pi}{4}) \cdot (\frac{\pi}{2})^2$, since the points y run from (what we might denote as) $x - \pi$ to $x + \pi$. □

3.2 Bounding Far(K)

3.2.1 Introduction

As in the previous case, we shall bound the inner integral,

$$\int_{\mathrm{arc(x,y)}\geq\pi} \frac{|< T_x\,, T_y\,, x-y >|}{|x-y|^3} \quad ,$$

then multiply by L to bound the double integral. We seek a bound of the form (constant independent of K) $\cdot$ (total curvature of K).

As before, we write the integrand as the triple product of three unit vectors, divided by $|x-y|^2$. Since the numerator has magnitude at most 1, it suffices to bound

$$\int_{\mathrm{arc(x,y)}\geq\pi} \frac{1}{|x-y|^2} \quad .$$

For any smooth curve with thickness radius R, it is shown in [9] that points x, y with $\mathrm{arc}(x,y) \geq \pi R$ must have $|x-y| \geq 2R$. So in our situaton, when $\mathrm{arc}(x,y) \geq \pi$, we know $|x-y| \geq 2$. The proof will then proceed along the following lines: We ask how much length of K can lie within 2 to 3 units of point x? within 3 to 4? etc. From Lemma 1.1, the answer has something to do with the total curvature of K. If the part of K that lies in a given spherical shell about x were connected, we could apply the lemma directly. However, we expect K to oscillate, go far from x, then back closer, then farther, etc. We will use Lemma 1.2 to account for this, though we need some extra complications to properly handle small oscillations. We would like to know that the total length of K that lies in each spherical shell about x with radii [n, n+1] is at most proportional to $\sqrt{n}$ (where the constant of proportionality will have a factor $\kappa(K)$). If that were guaranteed, we would have a bound of the form

$$\int_{\mathrm{arc(x,y)}\geq\pi} \frac{1}{|x-y|^2} \;\leq\; \text{(universal constant)} \cdot \kappa(K) \cdot \sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^2} \; .$$

But a given knot K might not have such uniformity from shell to shell. We need to take a more indirect route: We first bound the total length of K that can lie in any ball of given radius n about x. We then find an upper bound for the sum

$$\sum \frac{\text{length in shell } [n, n+1]}{n^2} \tag{23}$$

by comparing the actual spatial distribution of length of K relative to x to a hypothetical distribution that would maximize this sum.

3.2.2 Notation

Let Y denote the smooth curve $\{y \in K \mid \text{arc}(x, y) \geq \pi\}$. This is the domain of integration for the inner integral of $\text{Far}(K)$. For any $a > 0$, let S_a denote the sphere about x of radius a, and for $a < b$, let $S[a, b]$ denote the spherical shell bounded by S_a and S_b.

3.2.3 Cut Y into small pieces

We want to bound the measure of each set $Y \cap S[n, n+1]$, $n = 2, 3, \ldots \infty$. Our task is complicated by the fact that $Y \cap S[n, n+1]$ typically will not be connected, and may involve a large number of short components. To handle this, we discretize the problem and use a counting argument to provide the desired bounds.

Pick any integer $M >$ arc length of Y, and cut Y into M consecutive arcs of equal length. Let ϵ denote the length of each sub-arc, and note $\epsilon < 1$. The particular value of M will not matter for our analysis; we end up bounding infinite sums. What matters is that each arc is shorter than the distance between consecutive spheres, and yet a string of many (relative to epsilon) such arcs represents a guaranteed amount of length of Y.

We assign to each arc Y_i a label $1, 2, 3, \ldots$ representing approximately the shell $S[n, n+1]$ that contains Y_i. Specifically, if $Y_i \subset S[n, n+1]$, assign label n. If Y_i is not entirely contained in one shell, then it must intersect a sphere $S[n]$. Because $\epsilon < 1$, an arc Y_i can intersect at most one sphere $S[n]$, and we assign that label n to the arc. Note that if an arc Y_i carries label n, then $Y_i \subset S[n, n+1+\epsilon]$ and $Y_i \cap S[n, n+1] \neq \emptyset$.

3.2.4 Discretize the problem

For each integer n, let $\phi(n)$ denote the total number of arcs Y_i that are labeled n. Thus

$$\int_Y \frac{1}{|x-y|^2} = \sum_i \int_{Y_i} \frac{1}{|x-y|^2} \;\leq\; \sum_n \phi(n) \frac{\epsilon}{n^2} \,. \tag{24}$$

We would like to say that the numbers $\phi(n)$ are bounded by something on the order of $\kappa(K)(1/\epsilon)\sqrt{n}$. This is not quite true, however, since, for a given knot K, and particular n, the value of $\phi(n)$ can be very large. But we shall construct an upper bound for the whole sum in (24) in which the summands are more predictable. To do this, we need the auxiliary function that counts the total number of arcs that are labeled between 2 and n. Define

$$\Phi(n) = \sum_{j=2}^{n} \phi(j) \; . \tag{25}$$

We proceed as follows:

1. Abstract the arc Y as the string of integers $\mathcal{L} =< a_1, a_2, \ldots, a_m >$, where a_i is the shell label of Y_i.

2. Use the bound $\kappa(K)$ on total curvature of Y to determine constraints on the symbol string $\mathcal{L}$ that ultimately give bounds on the values $\Phi(n)$.

3. Note that the functions ϕ and Φ make sense for any string $\mathcal{L}$ of m integers

4. For any finite string $\mathcal{L}$ of integers, define an "energy"

$$E(\mathcal{L}) = \sum \phi(n) \frac{\epsilon}{n^2} \; . \tag{26}$$

5. Consider all strings $\mathcal{L}$ that satisfy the constraints on Φ determined by our Y. Maximize $E(\mathcal{L})$ over this set of strings by finding a string $\mathcal{L}^*$ for which the energy is maximum.

6. Find a bound for the value $E(\mathcal{L}^*)$, which is an upper bound for the final sum in (24), of the form we want.

The first step is to translate the curvature bound for Y into constraints on $\mathcal{L}$, the string of symbols. To shorten formulas, use κ to denote $\kappa(K)$.

For each n, there cannot be too many *consecutive* symbols $\leq n$. A string of q consecutive symbols $\leq n$ represents a connected arc $A \subset Y$ of length $q\epsilon$ contained in the ball $S[0, n+1+\epsilon]$. If q is such that $q\epsilon \geq 3(n+1+\epsilon)$, then, by Lemma 1.1, $\kappa(A) \geq 1/3$. Thus, for such q, we can have no more than 3κ such strings. So the first constraint is:

- For each n, the string $\mathcal{L}$ contains at most 3κ pairwise disjoint substrings of length $\frac{3(n+1+\epsilon)}{\epsilon}$ consisting of integers $\leq n$.

We obtain a second constraint, this time on "jumps". If, in the string $\mathcal{L}$, we observe the sequence $\langle \dots n \dots n+1 \dots n \dots \rangle$, we cannot infer any particular curvature contribution. But if we see $\langle \dots n \dots n+2 \dots n \dots \rangle$, then we can. Let us call a substring $\langle n \dots n+2 \dots n \rangle$ of $\mathcal{L}$ a *jump at level* n.

An arc Y_i with label n has nonempty intersection with $S[n, n+1]$; an arc with label $n+2$ is contained in $S[n+2, n+3+\epsilon]$. Thus if $\mathcal{L}$ includes a jump at level n, then we can infer the existence of a subarc A of Y (the closure of a component of $Y \cap S(n+1, \infty)$) that starts at the sphere S[n+1] and reaches as far out as some $S[b]$, $b \geq n+2$, before heading back to end at $S[n+1]$. By picking an "innermost" such arc, we can ensure precisely the situation of an arc contained in $S[n+1, b]$ that has both of its endpoints on $S[n+1]$ and some interior point on $S[b]$, where $b \geq n+2$. By Lemma 1.2, such an arc contributes more than $1.2\frac{1}{\sqrt{n+1}}$ to total curvature. Thus we have our second constraint:

- For each n, the string $\mathcal{L}$ cannot have $\frac{1}{1.2}\,\kappa\,\sqrt{n+1}$ pairwise disjoint jumps of level n.

3.2.5 Count and complete the proof

We now combine the constraints on substrings and jumps in $\mathcal{L}$ to get bounds on $\Phi(n)$.

Proposition 1.1. *For each* $n \geq 2$,

$$\Phi(n) < 24\,\frac{\kappa}{\epsilon}\,n^{3/2}. \tag{27}$$

Proof. Suppose, to the contrary, that for some n, $\mathcal{L}$ does have that many symbols $2, 3, \dots, n$. The bound $24n^{3/2}$ was chosen to be a simple expression such that, for $n \geq 2$ and $\epsilon < 1$,

$$24n^{3/2} > \left(3 + \frac{1}{1.2}\sqrt{n+1}\right)\,\left(3(n+2+\epsilon)\right). \tag{28}$$

Note we are using $(n+2+\epsilon)$, which comes from the first constraint for $n+1$, not for n. This is connected to our definition of "jump at level n".

Visualize the string $\mathcal{L}$ so that, temporarily, only the symbols $2, 3, \dots, n$ are visible. Parse these into pairwise disjoint substrings of length $\frac{3(n+2+\epsilon)}{\epsilon}$. By assumption, we have (considerably) more than 3κ of these substrings. So in the actual string $\mathcal{L}$, a number of these substrings must get broken up by inserted symbols $\geq n+1$. Now make all the symbols $a_i = n+1$ in $\mathcal{L}$ visible as well. These certainly can break up substrings consisting only of symbols $2, 3, \ldots, n$, but they offer no improvement on our situation of exceeding the total curvature bound: we chose the lengths of the substrings to be long enough that even if they were made from symbols $2, 3, \ldots, n+1$, they would still each be contributing $\geq (1/3)$ to total curvature, so we cannot have more than 3κ of these. Thus we must have some symbols $\geq n+1$ in the original string $\mathcal{L}$ to break up a number of the "offending" substrings. How many of the substrings can be broken by inserting symbols $a_j \geq n+2$? Each offending substring that gets broken this way represents at least one jump at level n. So we must have have fewer than $\frac{1}{1.2}\,\kappa\,\sqrt{n+1}$ such interruptions. But we chose the original number of symbols $2, 3, \ldots, n$ large enough that the number of offending substrings is greater than the number of possible interruptions plus the maxumum number we could tolerate to be uninterrupted. We conclude that $\Phi(n)$ cannot be that large. □

We have completed Step 2 of our plan, and now proceed. The functions ϕ and Φ make sense for finite abstract strings $\mathcal{L}$:

$$\phi(n) = \text{number of symbols } a_i \text{ of } \mathcal{L} \text{ that are} = n \; ; \tag{29}$$

$$\Phi(n) = \textstyle\sum_{j=2}^{n} \phi(j) \; , \tag{30}$$

and we can define the "energy"

$$E(\mathcal{L}) = \sum \phi(n) \frac{\epsilon}{n^2} \; . \tag{31}$$

The maximum shell label occurring in the string $\mathcal{L}$ associated to Y certainly is less than the total number of subarcs, M. Consider the set $\mathfrak{S}$ of all abstract strings of integers $2, 3, \ldots, M$ for which the functions Φ satisfy Proposition 1.1. We shall find a bound for the maximum of $E(\mathcal{L})$ over all $\mathcal{L} \in \mathfrak{S}$, so this bound also will be a bound for the integral over Y in the inequality (24). The method is to construct an abstract string $\mathcal{L}^*$ that satisfies the constraint (27), so it is an element of $\mathfrak{S}$, and has maximum possible $E(\mathcal{L})$.

If we take a string in $\mathfrak{S}$ and change some symbol a_i to a lower integer, that increases $E(\mathcal{L})$. Also, if we introduce a new additional symbol a_j somewhere in $\mathcal{L}$, that increases $E(\mathcal{L})$. We cannot do this too much, since the resulting string would no longer lie in $\mathfrak{S}$. We construct the desired $\mathcal{L}^*$ by having each $\phi(n)$ be maximum, starting from $n = 2$ and working up.

For $n = 2$, Proposition 1.1 says $\Phi(2) < 24\,\frac{\kappa}{\epsilon}\,2^{3/2}$. Since $\phi(2) = \Phi(2)$, we start $\mathcal{L}^*$ with exactly the maximum allowable number of 2's, that is

$$\phi(2) = \text{ greatest integer less than } 24\,\frac{\kappa}{\epsilon}\,2^{3/2}. \tag{32}$$

To keep the calculations simpler, we will use the bounds themselves, rather than the "greatest integer less than". This changes the ultimate coefficient in our theorem, but not the basic structure, in particular the exponents.

Proposition 1.1 next says that $\Phi(3) < 24\,\frac{\kappa}{\epsilon}\,3^{3/2}$. But $\Phi(3) = \Phi(2) + \phi(3)$. Having forced $\Phi(2)$ as above, we now make $\Phi(3)$ the maximum possible by choosing

$$\phi(3) = 24\,\frac{\kappa}{\epsilon}\,3^{3/2} - 24\,\frac{\kappa}{\epsilon}\,2^{3/2}. \tag{33}$$

Continuing inductively, we construct the string $\mathcal{L}^*$ so that for each $n = 2, 3, \ldots, M$,

$$\phi(n) = 24\,\frac{\kappa}{\epsilon}\,n^{3/2} - 24\,\frac{\kappa}{\epsilon}\,(n-1)^{3/2}. \tag{34}$$

We have now completed steps 1-5 of our plan, and just need to bound the energy $E(\mathcal{L}^*)$. We do this via the infinite sum, where $M \to \infty$.

Proposition 1.2.

$$\sum_{n=2}^{\infty} (\frac{\epsilon}{n^2}) \left(24\,\frac{\kappa}{\epsilon}\,n^{3/2} - 24\,\frac{\kappa}{\epsilon}\,(n-1)^{3/2} \right) \leq 66\,\kappa\,. \tag{35}$$

Proof. Cancel the ϵ's and factor out the common κ and coefficient 24 The resulting function,

$$f(n) = \frac{n^{3/2} - (n-1)^{3/2}}{n^2} \tag{36}$$

is summable over $n = 2 \ldots \infty$; the sum is less than the integral

$$\int_{x=1}^{\infty} \frac{x^{3/2} - (x-1)^{3/2}}{x^2} = \frac{3}{2}\pi - 2\,. \tag{37}$$

Multiplying this by 24, we get a constant < 66.

□

This completes the final step of our plan, and the proof of Theorem 1.

□ □ □

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