Why "Knot"?

by Gregory Buck

If you look about you as you are reading this, chances are you will see a knot somewhere: in your shoelaces, in your sweater, or perhaps in your hair. Knots are common things. You'll find knots anywhere a strand of nearly anything gets long enough. (Think of a garden hose or an extension cord.) Of course, there are different types of knots, different *patterns* of knotting.

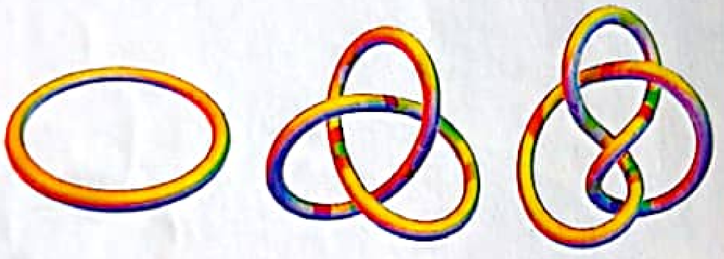


Figure 1

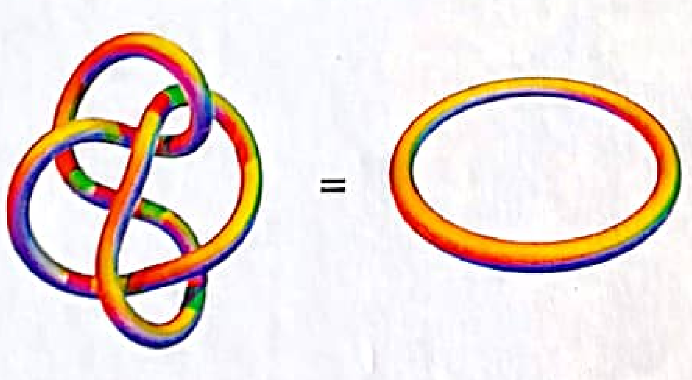


Figure 2

Mathematics is sometimes called the "science of patterns," so perhaps it won't surprise you to learn that there is a mathematics of knotting.

Let's consider what makes a *mathematical knot.* Tie a knot in a piece of string, and then join the open ends of the string, perhaps by taping them together. This is how mathematicians think of knots. The idea is that when the *loop* is closed, there is no way the knot can be taken out of the string.

Now, different ways of tying produce different knot types. In Figure 1 at left there are three knot types: the *unknot* (a circle), the *trefoil* (the simplest knot, so named because of its three-part shape), and the *figure eight* knot. But a knot does not always appear in its most recognizable form. Two knots are *equivalent* (of the same type) if one can be *deformed* (changed) to the other without breaking the string. Figure 2 is a picture of two equivalent knots.

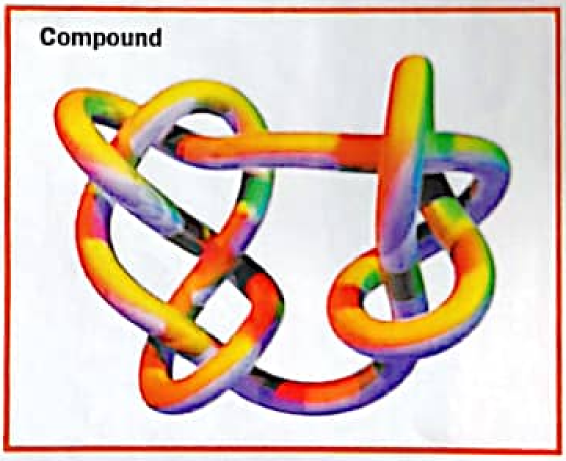
There is an *algebra* of knots, which means that we can add knots. We do this by tying two or more knots next to or on top of each other in the same loop. It has been proven that there are no *negative knots.* That is, you cannot add two knots together and end up with a knot equal to the circle, or unknot. But you might try it just the same!

Illustrations created by Robert Scharein using the computer program Knot Plot. *To see more knots visit the* Knot Plot *Web site at:* **http://www.cs.ubc.ca/nest/imager/contributions/scharein/KnotPlot.html**

Gregory Buck has some fun with nautical rope. Can you identify these knot types?

Gregory Buck appreciates the "craft" of knot making as well as the theory behind it. His office at St. Anselm College in Manchester, New Hampshire, is filled with knots in many forms.

Figure 3



When you add two knots, the result is a *compound knot* as shown in Figure 3. A knot that is not the sum of other knots is called a *prime knot*. Two of the most common methods for creating prime knots are pictured below. Figure 4 gives us variations of what is called a ***torus*** knot and Figure 5 gives us a member of the family of *twist knots*.

Torus — In mathematics, a surface having the shape of a doughnut

Another sort of algebra of knots arises when we look at braids. Consider the common three-strand braid, often used to braid hair. This braid clearly employs a pattern: right to the middle, left to the middle, right to the middle, and so on. Think of each of these crossovers as an individual move, and imagine that we give each move some symbols to represent it. For example, let's have strand 1 represent the right strand, strand 2 the middle strand, and strand 3 the left strand. Also, let "M" stand for "move." Let M(1,2) mean that we cross strands 1 and 2, with 1 going over 2. After this move we have a new strand 1 and a new strand 2, but strand 3 is the same. In this way the usual three-strand braid could be written: M(1,2), M(3,2), M(1,2), M(3,2), and so on (see Figure 6). Notice that if you do the combination M(1,2), M(2,1), the result is that strands 1 and 2 are not tangled, but with the combination M(1,2), M(1,2), the first two strands are twisted about each other. So we can write that M(1,2) + M(2,1) = 0, but M(1,2)

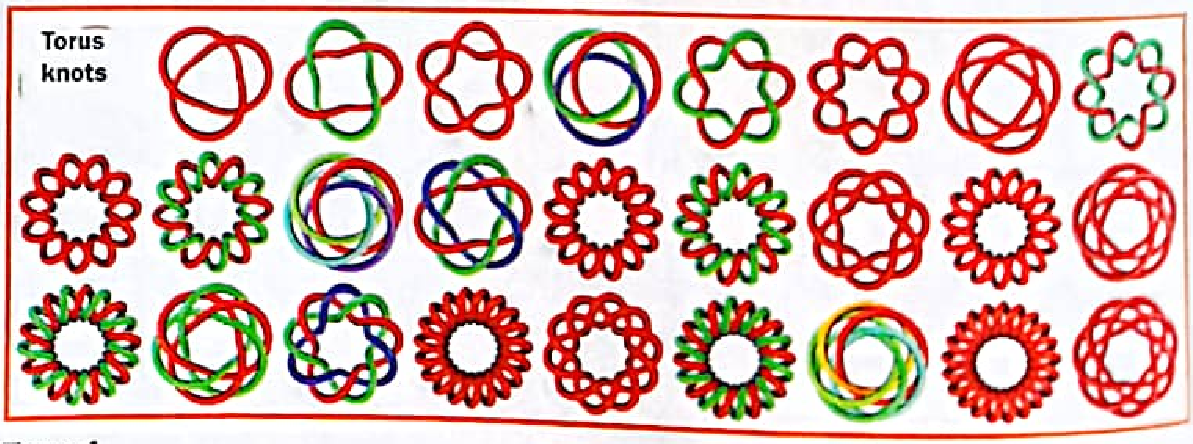


Figure 4

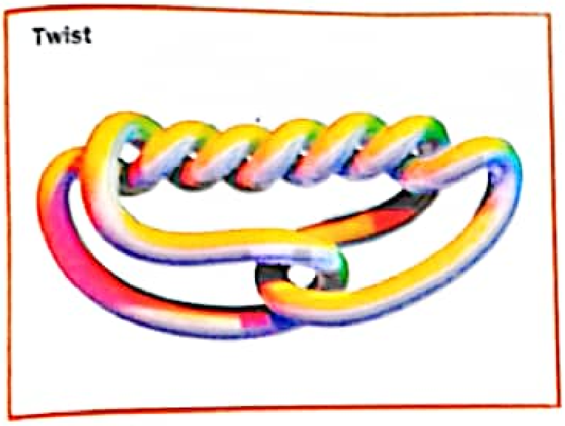


Figure 5

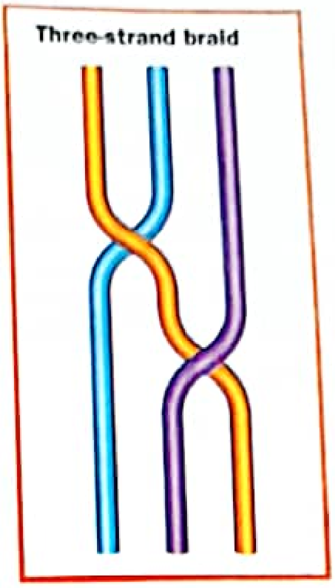


Figure 6

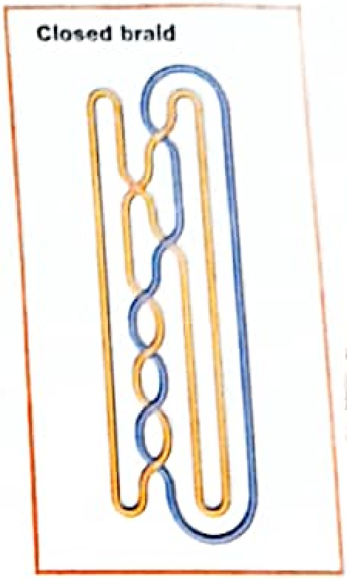


Figure 7

$+ M(1,2) \neq 0$.

A *closed braid* comes about by joining the two sets of open ends. The strands are joined in order, so, for example, the leftmost strand at the bottom of the braid is joined with the leftmost strand at the top. But note that these are not necessarily, nor even usually, the same strand. That is, if you trace through the braid the strand that begins on the left at the top of the braid, you may find that at the bottom it is at, say, the middle position. So, when the braid is closed that strand connects with the middle strand at the top. A closed braid gives us either a knot, if the result is one long strand, or a *link,* if the result is two or more closed loops (see Figure 7).

When we classify knots we often rank them by complexity. There are several different methods for measuring the complexity of a knot. One is to count how many different *crossings* are necessary to draw the knot on paper. It takes at least three crossings to draw the trefoil. The figure eight requires four crossings. Another method for measuring the complexity of a knot is to measure how long a piece of rope is required to tie the knot. (Remember, the knot is finished by joining the two open ends to make a loop.) Also, the length required is affected by the thickness, or *diameter,* of the rope. The thicker the rope, the longer it needs to be to make a knot. To account for this, we divide the required length of the rope by the rope's diameter.

More Than Rope

Knot theory has lots of applications, some on the smallest scale (such as in subatomic physics), and some on a very large scale (for instance, the solar corona has long, tangled magnetic loops that we can sometimes observe with special instruments from Earth). Recently knot theory has found an interesting application in microbiology.

DNA is thought of as the basis of life, because it carries the plan of how an organism is to grow and develop. The DNA molecule is a very long, thin strand that is almost always knotted in its natural state. Because life proceeds by DNA replication, the molecule has to make copies of itself. For this to happen the strand of the molecule must split apart. But the knotted state of the molecule makes this replication process extremely difficult, because the molecule divides down the middle, the long way.

Imagine a long, closed zipper that is knotted, with its head joining its tail, forming a loop. Now imagine unzipping the zipper, all the way around. You now have two strands, which themselves are

ABOVE: This computer-generated model illustrates the "knotty" nature of DNA.

linked. But the strands cannot remain permanently linked. They must find a way to unlink in order to separate. Because the strands are closed loops, there is only one possible way for them to come unlinked: Something must cut the strands, to let one strand pass through the other.

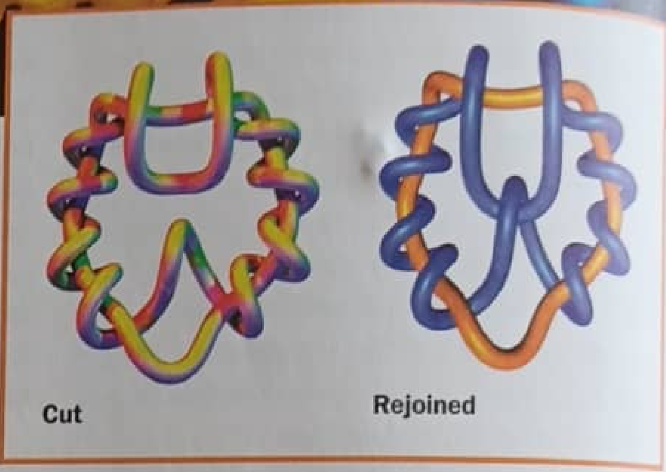


Figure 8

When scientists realized this they went looking for the substance that cuts the strands enabling the DNA molecule to replicate. They found an enzyme they have called *topoisomerase*. The name is a tongue twister, but it explains exactly what the enzyme does: It *erases* the ***topology***, the link within the DNA molecule. Topoisomerase literally cuts one strand of the DNA molecule, passes the other strand through it, and then rejoins the cut strand as shown in Figure 8. A current topic of research in mathematical knot theory is the effort to understand just how many of these cuts need to be made to unlink a typical DNA molecule. There are lots of questions in knot theory we haven't yet found the answers to. Maybe in the future *you* can help answer some of these "knotty" questions!

Topology – The anatomical structure of a part of the body, in this case the DNA molecule

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