

Four-thirds power law for knots and links

Physical knot theory has recently been applied to polymer dynamics, and specifically to gel electrophoresis of DNA^{1,2}. Knot energies³⁻⁶ measure the complexity of a knot conformation; minimum energy conformations are considered canonical or 'ideal' conformations. The rope length of a knot is one such measure of energy⁶, and an approximately linear relationship between rope length and the average crossing number for minimum rope-length conformations of simple knots has been reported⁷. Here I show that a linear relationship cannot hold in general: the rope length required to tie an N -crossing knot or link varies at least between $\sim N^{3/4}$ and $\sim N$.

Consider four measures of knot conformation complexity. Imagine a solid disc of radius R centred at x and normal to K at each point x along the parametrized smooth knot K . $R(K)$ is the largest R so that the disks are disjoint. The rope length is $L(K)=[\text{arclength}(K)]/[R(K)]$. The crossing number $C(K)$ of the knot-type K is the necessary number of crossings in the planar diagram of K .

For points x, y on K , let ρ denote $|x-y|$, \mathbf{r} the vector $(x-y)/|x-y|$, and $d\mathbf{x}$ a line element of K . The average crossing number is given by

$$A(K) = (1/4\pi) \iint_{K \times K} (|\mathbf{dx}, \mathbf{dy}, \mathbf{r}|) / \rho^2$$

where the integrand numerator is the positive triple scalar product of the three vectors. This integral gives the average number of crossings of K , when viewed from an arbitrary perspective. The symmetric energy³ is given by

$$S(K) = \iint_{K \times K} (|\mathbf{dx} \times \mathbf{r}| |\mathbf{dy} \times \mathbf{r}|) / \rho^2$$

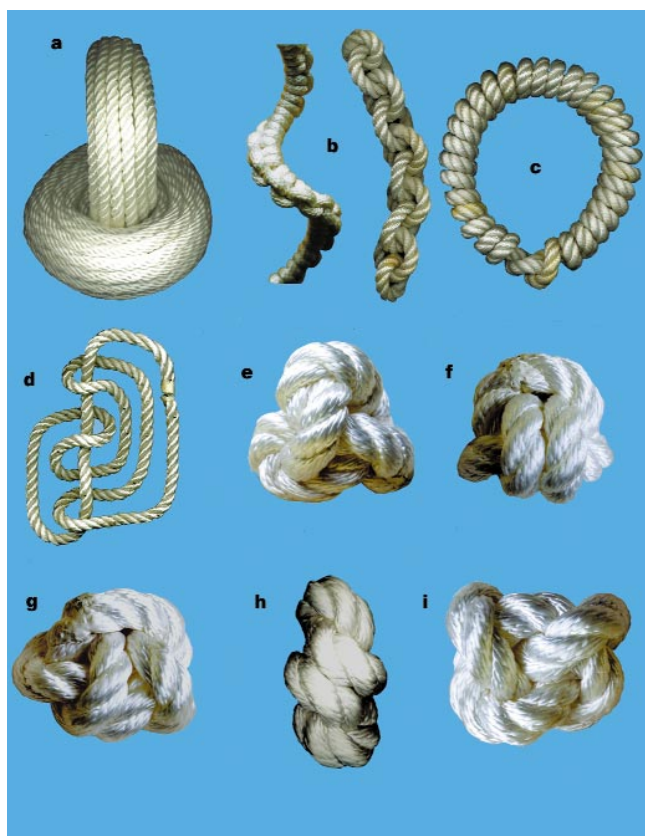
If K is the centre curve of a radiating tube, T , with relatively small radius, which is relatively far from self-intersection, then $S(K)$ measures the self-radiation of the tube.

A natural 4/3 power law relates rope length to measures based on inverse-square laws (including the crossing and writhe numbers, and $S(K)$), and a linear bound relates $S(K)$ to A . We can show^{4,17} that

$$11L(K)^{4/3} \geq S(K) \geq 4\pi A(K) \geq C(K)$$

Imagine a very long piece of rope packed as tightly as possible in a roughly spherical

Figure 1 Knot conformations. **a**, Packed Hopf tori. **b**, 'Linear' conformations. Left, a product of trefoils; right, a thick chain with a linear relationship between crossing number and rope length. The chain also seems to be a continuous family of minima for rope length, in which case minima are not isolated in the link class. **c**, 'Linear' conformation of a twist knot – apparent minimum. **d**, An N -crossing knot fits in a square of side order N . **e, f**, Minima for figure-eight and square knot respectively. No particular accuracy is claimed – these knots were tied before the calculation of the computer data⁷, and both the conformation and the values for rope length match almost exactly, as did several other knots. **g**, Minimum for the 'granny' knot, differing in shape from the minimum found by computer⁸, and having a different symmetry. **h**, Another view. We estimate this to be the true global minimum. **i**, Some support for the accuracy of the rope calculations is given by this conformation of the five-crossing torus knot, which by rope seemed to be a lower minimum than the conformation reported in ref. 7. Further computation confirmed this.



shape, scaled so that $R(K)=1$, with total length L . The inverse-square 'energy' ($S(K)$, A , writhe, and so on) can be estimated by assuming the 'mass' of the knot is concentrated at points p on the integer lattice. Concentric shells of unit thickness about each p each contribute the same amount, so the contribution for p is that constant multiplied by the number of shells, which is of the order of $L^{1/3}$. Multiplying by the number of points, L , gives $L^{4/3}$. The proof that $S(K)$ linearly bounds A is simple vector geometry.

The $4/3$ exponent is sharp. Consider the Hopf link of two tori in its natural geometrical position. Fill each torus with N loops parallel to the centre curve, each loop a strand of radius 1 (Fig. 1a). Then with any tight packing of the loops, the minor radii of the tori is of the order of \sqrt{N} . The conformation fits inside a sphere of radius $4\sqrt{N}$, so the total rope length is about $N^{3/2}$. Each loop is linked with N loops in the perpendicular torus, so the crossing number is about N^2 . Therefore the rope length is of the order of $C(K)^{3/4}$. Because $11L(K)^{4/3} \geq 4\pi A(K) \geq C(K)$, this example has A in the order of $L(K)^{4/3}$.

The minimum rope length for a knot is bounded by $3C(K)^2$. This can be seen by arranging the knot so that the crossings are evenly spaced along a line (Fig. 1d). For the simpler knot types, $L(K)$, $S(K)$ and A in minimized conformations all 'appear' to be

linearly related⁷. An explanation is that the simpler conformations are 'planar': from most perspectives a unit arc of the knot crosses only a few other unit arcs.

As complexity increases, there are many families of knots and links with three-dimensional growth, exhibiting the $4/3$ power law. Families with single-dimensional growth (Fig. 1b,c) have a linear relationship among the measures. With planar growth, we expect A to be linear with $C(K)$ and $S(K)$ to be of the order of $L(K)\log L(K)$.

We propose that the rope length required (Fig. 1e–i) to tie an N -crossing knot or link varies only between $k_1 N^{3/4}$ and $k_2 N$. Other investigators have also recently observed the $4/3$ law in knots on the cubic lattice⁹ and in vector fields¹⁰.

A good knot energy has only a finite number of knot types realized below any given energy level. Our theorem gives us this property for $L(K)$ and $S(K)$, proving that there is a finite number of knots that can be tied with a finite length of mathematical rope.

Gregory Buck

Department of Mathematics, Saint Anselm College, Manchester New Hampshire 03102, USA
email: gbuck@anselm.edu

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Heartbeat synchronized with ventilation

It is widely accepted that cardiac and respiratory rhythms in humans are unsynchronized¹. However, a newly developed data analysis technique allows any interaction that does occur in even weakly coupled complex systems to be observed. Using this technique, we found long periods of hidden cardiorespiratory synchronization, lasting up to 20 minutes, during spontaneous breathing at rest.

Synchronization is a universal phenomenon that occurs when two or more nonlinear oscillators are coupled. It is observed in many fields of science and is widely applied in engineering. The case of synchronization in periodic, or even noisy, oscillators is well understood^{2–4}. The notion of synchronization has often been used to analyse the interaction between physiological (sub)systems¹, but these studies have been restricted to almost periodic rhythms. No approach has been suggested to probe the weak interactions between such irregular and non-stationary oscillators as the human heart and respiratory system.

These two physiological systems are known to be coupled by several mechanisms, but apart from respiratory modulation of heart rate, first observed in 1847 and known as 'respiratory sinus arrhythmia' (RSA)^{5–7}, no other effects have been reported. Moreover, in spite of some early communications⁸, it has been concluded that "there is comparatively weak coupling between respiration and the cardiac rhythm, and the resulting rhythms are generally not phase locked"¹.

We used the concept of phase synchronization of chaotic oscillators^{9,10} to develop a technique to analyse irregular non-stationary bivariate data. We analysed data obtained in non-invasive examinations of eight healthy volunteers (14–17-year-old, high-performance swimmers; four of them male and four female). While subjects lay at rest, electrocardiograms (ECGs) were recorded while respiratory